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of the twisted Yangian

LETTER TO THE EDITOR

Resolution of the $GL(3) \supset O(3)$ state labelling problem via the $O(3)$-invariant Bethe subalgebra of the twisted Yangian

P D Jarvis[1] **and R B Zhang**[2]

[1] School of Mathematics and Physics, University of Tasmania, GPO Box 252–21, Hobart, TAS 7001, Australia
[2] School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail: Peter.Jarvis@utas.edu.au and rzhang@maths.usyd.edu.au

Abstract
The labelling of states of irreducible representations of $GL(3)$ in an $O(3)$ basis is well known to require the addition of a single $O(3)$-invariant operator, to the standard diagonalizable set of Casimir operators in the subgroup chain $GL(3) \supset O(3) \supset O(2)$. Moreover, this 'missing label' operator must be a function of the two independent cubic and quartic invariants which can be constructed in terms of the angular momentum vector and the quadrupole tensor. It is pointed out that there is a unique (in a well-defined sense) combination of these which belongs to the $O(3)$-invariant Bethe subalgebra of the twisted Yangian $Y(GL(3); O(3))$ in the enveloping algebra of $GL(3)$.

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The necessity for using adapted bases compatible with kinematical or dynamical symmetries of a quantum system has long been recognized as an essential tool for dealing with the implications of the symmetries in terms of constraints on physical quantities, and for reducing the number of computations. Unfortunately, the symmetries of interest in physics are not always those which admit 'canonical' bases, possessing complete sets of orthonormal basis functions which are common eigenfunctions of a maximal set of commuting Casimir operators, in a chain of subgroups extending from the group of interest up to the maximal (unitary) group of transformations on the space in question. A case of a serendipitous group labelling occurred in Racah's early work on equivalent f electrons for rare earth spectra [12] where the exceptional group G_2 was found to extend the symmetry adapted group–subgroup chain from $U(7) \supset O(7) \supset O(3)$ to $U(7) \supset O(7) \supset G_2 \supset O(3)$, in a way which made the specifications of the electronic wavefunctions complete for the cases studied. In fact all steps in this chain, and generically $U(N) \supset O(N)$, are of the type where there are one

or more 'missing labels'—in order to resolve multiplicities in the restriction of irreducible representations of the larger group to the subgroup, the degeneracies should be removed with the help of additional subgroup invariant operators which cannot be Casimir operators of the group or the subgroup, but must be invariant operators taken in the enveloping algebra of the group that commute with each other. It follows from general arguments based on the double commutant theorem that there always exist enough such invariant operators to provide the missing labels for resolving multiplicities.

In this note we study the case of $N = 3$, namely the classic '$U(3) \supset O(3)$ state labelling problem', which is ubiquitous in atomic, nuclear and many-body physics. Below, we give a brief introduction to the notation necessary to define the problem, and we review the known result that there are two admissible additional $O(3)$ invariant operators in the enveloping algebra of $U(3)$ which are candidates for the single 'missing label' needed in this case. We then turn to the formalism of Yangian algebras, which provide a powerful way of handling the (infinite dimensional) enveloping algebras of the classical Lie algebras. Specifically, we consider the so-called Bethe subalgebras [9, 10], which are maximal commutative subalgebras. For the $N = 3$ case, we show explicitly that there is a *unique* (in a well-defined sense) combination of the candidate $O(3)$ non-subgroup invariants, which belongs to the $O(3)$-invariant Bethe subalgebra of the twisted Yangian $Y(GL(3); O(3))$.

The $SU(3) \supset O(3)$ state labelling problem is comprehensively examined in [11], which also includes extensive numerical evaluations for low-dimensional representations. In the following we work with the complex algebras, and so refer to $GL(3)$, $O(3)$ and so on (as well as following the physics convention of not distinguishing notationally between group and algebra).

Consider the standard generators E_{ij}, $i, j = 1, 2, 3$ of $GL(3)$, with commutation relations modelled on those of the defining 3×3 matrix units e_{ij}, acting on three basis vectors e_i in the usual way:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \tag{1}$$

For applications in many-body physics the orbital angular momentum generators are given by

$$L_{ij} = E_{ij} - E_{ji} \tag{2}$$

from which the usual vector angular momentum generators follow as $L_1 = L_{23}$, $L_2 = L_{31}$, $L_3 = L_{12}$. For labelling states in irreducible representations of $GL(3)$ one requires a maximal set of commuting operators. Those in the group–subgroup chain $GL(3) \supset O(3) \supset O(2)$ are the associated Casimir operators. Taking them to be the standard Gel'fand invariants gives for $GL(3)$, at increasing degree,

$$C^{(1)} = \sum_{i=1}^{3} E_{ii}, \qquad C^{(2)} = \sum_{i,j=1}^{3} E_{ij} E_{ji},$$

and

$$C^{(3)} = \sum_{i,j,k=1}^{3} E_{ij} E_{jk} E_{ki}, \tag{3}$$

the linear Casimir being of course the number operator $N \equiv E_{11} + E_{22} + E_{33}$ (which determines the energy $\left(N + \frac{3}{2}\right)\hbar\omega$ if the system is a three-dimensional isotropic oscillator). For ease of writing, we adopt the notation $\langle E \rangle$, $\langle E^2 \rangle$ and $\langle E^3 \rangle$ for these Casimir operators. Further invariants are the quadratic Casimir for $O(3)$,

$$C^{[2]} = \sum_{i,j=1}^{3} L_{ij} L_{ji}, \tag{4}$$

which we denote by $\langle L^2 \rangle$, and the $O(2)$ angular momentum component L_{12} (the Casimir operator being $|L_{12}|$).

The remaining $GL(3)$ generators are the quadrupole tensor

$$Q_{ij} = E_{ij} + E_{ji}, \qquad \text{or} \qquad Q'_{ij} = E_{ij} + E_{ji} - \tfrac{2}{3} N \delta_{ij}, \tag{5}$$

where the traceless form $Q' \in SL(3)$ separates the number operator N. It is proven in [11] using methods of invariant theory (or so-called 'integrity bases') that the algebraically independent $O(3)$ invariants in the $GL(3)$ enveloping algebra are at degrees 3, 4 and 6 and can be taken to be

$$X^{(3)} = \langle L Q' L \rangle = \sum_{i,j,k=1}^{3} L_{ij} Q'_{jk} L_{ki}, \qquad X^{(4)} = \langle L Q'^2 L \rangle = \sum_{i,j,k,l=1}^{3} L_{ij} Q'_{jk} Q'_{kl} L_{li},$$

and

$$X^{(6)} = \sum_{i,j,k,l,m,n=1}^{3} L_{ij} L_{kl} L_{mn} Q'_{ik} Q'_{lm} Q'_{jn}. \tag{6}$$

The operators $X^{(3)}$ and $X^{(4)}$ also appeared in works of Bargmann and Moshinsky [1, 2] on N particle systems (in the case $N = 3$) which were $O(3)$ invariant and with an additional $U(N)$ symmetry. Eigenvalues and other properties of the operators were investigated very extensively by these authors. We were also informed by an anonymous referee that the cubic operator played an important role in the computation of Clebsch–Gordan coefficients in the reduction $GL(3) \supset O(3)$ in an unpublished paper by Tolstoy (1985).

Observe the fact that $X^{(3)}$ and $X^{(4)}$ are primary invariants, while $X^{(6)}$ is secondary in that $(X^{(6)})^2$ is a polynomial in $X^{(3)}$, $X^{(4)}$ and terms of lower degree (the commutator $[X^{(3)}, X^{(4)}]$ gives essentially $X^{(6)}$ up to invariants of lower degree). Let $Z(GL(3); O(3))$ be the commutative subalgebra of $U(GL(3))$ generated by the Casimir operators of $GL(3)$ and $O(3)$, namely, the operators $\langle E \rangle$, $\langle E^2 \rangle$, $\langle E^3 \rangle$ and $\langle L^2 \rangle$. Then *any* $O(3)$-invariant operator that can be used for state labelling must be of the form $f(X^{(3)}, X^{(4)}) + X^{(6)} g(X^{(3)}, X^{(4)})$ for some polynomials f, g with coefficients in $Z(GL(3); O(3))$. As the object of interest is the restriction of an irreducible $GL(3)$ representation to $O(3)$, the $GL(3)$ Casimir operators, all being scalar multiples of the identity, lend no help for resolving multiplicities of $O(3)$ irreducible representations. It is the $O(3)$ Casimir and invariant operators such as $X^{(3)}$ and $X^{(4)}$ that provide the desired information for state labelling. In [11], explicit numerical evaluations of $X^{(3)}$ and $X^{(4)}$ are tabulated for irreducible representations of $SU(3)$ of high enough dimensions that multiplicities up to 3 occur in the restriction to $SO(3)$.

The infinite-dimensional Yangian algebras have been intensively studied in relation with applications to integrable systems and the inverse scattering method. They have a very remarkable formulation as noncommutative matrices over the ring of formal Laurent series, enabling the combinatorics of the coefficients involved in commutation relations and other constructs such as invariants and coproducts, to be handled by appropriate shifts of the formal variable u. A significant identification is that of the generators of the Yangian $Y(GL(N))$ with elements of the enveloping algebra of $GL(N)$. Denoting the generators of the Yangian by $t^{(m)}{}_{ij}$, $m = 0, 1, 2, \ldots$, this entails

$$T_{ij}(u) = \sum_{m=0}^{\infty} (-1)^m \frac{t^{(m)}{}_{ij}}{u^m}, \qquad t^{(m)}{}_{ij} = (E^m)_{ij}, \qquad \text{where}$$

$$\tag{7}$$

$$(E^0)_{ij} = \delta_{ij}, \quad \text{and} \quad (E^m)_{ij} = \sum_{k=1}^{N} E_{ik}(E^{m-1})_{kj} \quad \text{for} \quad m > 0.$$

More simply, the *inverse* of this series provides the evaluation homomorphism

$$T_{ij}(u) \rightarrow \delta_{ij} + \frac{E_{ij}}{u}. \tag{8}$$

from $Y(GL(N))$ to the universal enveloping algebra $U(GL(N))$. The generators are succinctly written with the Laurent series in matrix form

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes T_{ij}(u) \tag{9}$$

regarded as an $N \times N$ matrix with entries in the Yangian, or an element of $\text{End}(\mathbb{C}^N) \otimes Y(GL(N))[[u]]$. Central to such manipulations is the R-matrix (an operator on $\mathbb{C}^N \otimes \mathbb{C}^N$),

$$R(u) = u \cdot 1 + \mathbb{P}, \tag{10}$$

where $\mathbb{P}$ is the permutation operator defined by $\mathbb{P}e_i \otimes e_j = e_j \otimes e_i$. In terms of elementary matrices

$$\mathbb{P} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}. \tag{11}$$

The $O(N)$ twisted Yangian which we denote by $Y(GL(N); O(N))$ with generators $S(u)_{ij} \in Y(GL(N))[[u]]$ associated with the above embedding (2) of $O(N)$ in $GL(N)$ is introduced as

$$S_{ij}(u) = \sum_{m=0}^{\infty} \frac{s^{(m)}{}_{ij}}{u^m}, \qquad S(u) = \sum_{i,j=1}^{N} e_{ij} \otimes S_{ij}(u),$$

$$S(u) = T(u)\tilde{T}(-u) = 1 + \frac{E - \tilde{E}}{u} - \frac{E\tilde{E}}{u^2}, \tag{12}$$

using definition (9) above for $T(u)$ and with $\tilde{T}_{ij}(u) = T_{ji}(u)$. Here, $E = \sum e_{ij} \otimes E_{ij}$ and $\tilde{E} = \sum e_{ij} \otimes E_{ji}$. The relevant R-matrix is now the partial transpose

$$\tilde{R}(u) = u \cdot 1 + \mathbb{Q}, \tag{13}$$

where $\mathbb{Q}$ is the projection operator onto the one-dimensional $O(N)$-submodule $\mathbb{C} \sum_i e_i \otimes e_i$. In terms of elementary matrices

$$\mathbb{Q} = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}. \tag{14}$$

As with the $GL(N)$ Yangian, the commutation relations can be succinctly expressed using the $\tilde{R}$-matrix, and many structural properties of the algebra established (see [9, 10]; for even N the $Sp(N)$ twisted Yangian $Y(GL(N); Sp(N))$ can similarly be introduced).

One of the most fundamental aspects of the Yangian is the fact that the *trace* $\text{tr}[T(u)]$ of the Laurent series over $\text{End}(\mathbb{C}^N)$, namely,

$$\text{tr}[T(u)] = \sum_{i=1}^{N} T_{ii}(u), \tag{15}$$

commutes with $\text{tr}[T(v)]$ for arbitrary u, v—that is, the coefficients provide an infinite set of commuting operators. The diagonalizability of the transfer matrix is of course the underpinning of many of the applications of Yangians and the Yang–Baxter equation to integrable systems. The same property can also be proved for the trace of the twisted Yangian, $\text{tr}[S(u)]$.

The identification of Abelian subalgebras is not limited solely to the trace of the Yangian however. The so-called 'quantum determinant' is an object whose index structure is that of a determinant (of the Yangian matrix), but whose terms involve systematic shifts in the formal variable in order to compensate for the non-commutativity. The quantum determinant is thus a Laurent series in principle encoding an infinite number of coefficients, which this time belong to the *centre* of the algebra. Beyond the trace and the quantum determinant there is a remarkable set of infinite-dimensional Abelian subalgebras indexed by a fixed $N \times N$ matrix $Z = \sum_{i,j} e_{ij} z_{ij}$, the so-called Bethe subalgebras [9, 10] $B(GL(N); Z)$ and $B(O(N); Z)$. If Z is generic, in that it has a simple spectrum, these subalgebras are maximal, and generate the equivalent for the Yangians, of a complete set of commuting labelling operators for representations of finite-dimensional simple Lie algebras.

Henceforth, we specialize to the $GL(3) \supset O(3)$ case, and give the concrete constructions for the twisted Yangian $Y(GL(3); O(3))$. Because we are interested only in $O(3)$ invariants, we consider $Z = 1_{3\times3}$ and the Bethe subalgebra $B(O(3); 1_{3\times3})$. This choice of Z simplifies the general definitions, and the generators of the corresponding Bethe subalgebra are the coefficients in $u^{-1}, u^{-2}, \ldots$, of the following three elements $A_1(u), A_2(u), A_3(u)$:

$$A_1(u) = \mathrm{tr}_1[S_1(u-1)],$$
$$A_2(u) = \mathrm{tr}_{12}[\mathbb{A}_{12} \otimes 1 \cdot S_1(u-1)\tilde{R}_{12}(-2u+3)S_2(u-2)],$$
$$A_3(u) = \mathrm{tr}_{123}[\mathbb{A}_{123} \otimes 1 \cdot S_1(u-1)\tilde{R}_{12}(-2u+3)\tilde{R}_{13}(-2u+4)$$
$$\times S_2(u-2)\tilde{R}_{23}(-2u+5)S_3(u-3)]. \tag{16}$$

The subscript notation indicates to which of the subspaces various objects belong. For example in A_2, $S_2(u) = \sum_{i,j} 1 \otimes e_{ij} \otimes S_{ij}(u)$, whereas in A_3, $S_2(u) = \sum_{i,j} 1 \otimes e_{ij} \otimes 1 \otimes S_{ij}(u)$. The $\mathbb{A}$ are antisymmetrization operators acting on the appropriate spaces with

$$\mathbb{A}_{12} = 1 - \mathbb{P}_{12},$$
$$\mathbb{A}_{123} = 1 - \mathbb{P}_{12} - \mathbb{P}_{13} - \mathbb{P}_{23} + \mathbb{P}_{12}\mathbb{P}_{23} + \mathbb{P}_{13}\mathbb{P}_{23}. \tag{17}$$

Finally, the matrix objects are subjected to a total trace.

In $A_1(u)$ we recognize the basic transfer matrix trace discussed already, merely rewritten to emphasize its relationship to its partners in the Bethe subalgebra. Also, the top member $A_3(u)$ is the quantum determinant itself (always present, and independent of the matrix Z, because it is associated with the centre). In fact for the twisted Yangian, the quantum determinant is essentially the square of the quantum determinant for the Yangian itself. For the present $N = 3$ case, we can thus compute $A_3(u)$ via

$$A_3(u) = B_3(u)B_3(4-u), \qquad \text{where}$$
$$B_3(u) = \mathrm{tr}_{123}[\mathbb{A}_{123} \otimes 1 \cdot T_1(u-1)T_2(u-2)T_3(u-3)]. \tag{18}$$

From (17), (18) and the previous definitions it is straightforward to compute these Bethe subalgebra generators in terms of traces of polynomials in the $GL(3)$ and $O(3)$ generators as in (3) above. We find explicitly

$$A_1(u) = 3 - \frac{\langle E\tilde{E}\rangle}{(u-1)^2},$$
$$A_2'(u) = \langle L^2\rangle + \frac{\langle E\tilde{E}E\tilde{E}\rangle - \langle L^2\rangle}{(u-1)(u-2)},$$
$$B_3(u) = 6 - 2\langle E\rangle\left(\frac{1}{(u-1)} + \frac{1}{(u-2)} + \frac{1}{(u-3)}\right) - 3\frac{\langle E^2\rangle - \langle E\rangle^2}{(u-1)(u-3)}$$
$$- \frac{2\langle E^3\rangle + (\langle E\rangle + 1)(\langle E\rangle^2 - 3\langle E^2\rangle)}{(u-1)(u-2)(u-3)}. \tag{19}$$

where we have defined the essential part of $A_2(u)$ after making combinations with $A_1(u)$ by

$$A'_2(u) = (u-1)(u-2)\left(\frac{A_2(u)}{2u-3} + A_1(u)A_1(u-1) - A_1(u) - A_1(u-1) + 3\right). \qquad (20)$$

In simplifying the expressions basic symmetry properties have been used, for example $L_{ij} = -L_{ji}$ giving $\langle L \rangle = 0$, and from the definition of the angular momentum operators $\langle L^2 \rangle = 2\langle E^2 \rangle - 2\langle E\tilde{E} \rangle$. Similarly,

$$(E\tilde{E})_{ij} - (E\tilde{E})_{ji} = L_{ij} \qquad (21)$$

upon using the commutation relations (1), so that

$$\langle E\tilde{E}L \rangle = \langle LE\tilde{E} \rangle = \tfrac{1}{2}\langle L^2 \rangle. \qquad (22)$$

By taking appropriate linear combinations, the independent generators of $B(O(3); 1_{3\times 3})$ can be taken to be the set

$$\langle E \rangle^2, \ \langle E^2 \rangle, \ \langle E \rangle\langle E^3 \rangle, \ \langle L^2 \rangle, \quad \text{and} \quad \langle E\tilde{E}E\tilde{E} \rangle. \qquad (23)$$

A useful way to look at the Bethe subalgebra $B(O(3); 1_{3\times 3})$ is to consider it as an associative algebra over $B(O(3); 1_{3\times 3}) \cap Z(GL(3); O(3))$ generated by the operator $\langle E\tilde{E}E\tilde{E} \rangle$.

To complete the identification with labelling operators for $GL(3) \supset O(3)$, it is necessary to rewrite the operators (6) in the trace notation as above. In such expressions, use must again be made of the commutation relations in order to group similar terms. Within strings of the form $\langle X \rangle = \langle E^m \tilde{E}^n \cdots \rangle$, for example, simplifications that can be made are that the trace is cyclic in nature, and also that the transpose $\tilde{X}$ behaves (anti-)involutively (of course, $\langle \tilde{X} \rangle = \langle X \rangle$)—in both cases up to rearrangements in the order of terms, which produce invariants of lower degree after applying the commutation relations. It should also be noted that for $O(3)$, $\langle L^3 \rangle = \tfrac{1}{2}\langle L^2 \rangle$ and $\langle L^4 \rangle = \tfrac{1}{2}\langle L^2 \rangle(\langle L^2 \rangle + 2)$. Similarly, in $GL(3)$ the quartic Casimir $\langle E^4 \rangle = \text{tr}(E^4)$ is not algebraically independent, being of higher degree than the exponents for invariants of the group, namely 1, 2 and 3 for $GL(3)$. In this case by invoking the characteristic identity [7, 4] for the matrix E_{ij} (the analogue of the matrix Cayley–Hamilton identity, but with coefficients in the centre of the enveloping algebra), one can show that $\langle E^4 \rangle$ is a linear combination of $\langle E \rangle\langle E^3 \rangle$ and similar reducible terms of degree up to 4, consisting of products of traces with lower degree.

Because of the discussion following equation (6), we may consider, instead of $X^{(3)}$ and $X^{(4)}$ themselves, their combinations over $Z(GL(3); O(3))$ defined by

$$Y^{(3)} = X^{(3)} + \tfrac{2}{3}C^{(1)}C^{[2]},$$
$$Y^{(4)} = X^{(4)} + \tfrac{4}{3}C^{(1)}X^{(3)} + \tfrac{4}{9}(C^{(1)})^2 C^{[2]}.$$

Some very lengthy calculations yield

$$Y^{(3)} = -(\langle E\tilde{E}^2 \rangle + \langle \tilde{E}E^2 \rangle) + 2\langle E^3 \rangle - 3\langle E^2 \rangle + \langle E \rangle^2,$$
$$Y^{(4)} = -2\langle E\tilde{E}E\tilde{E} \rangle + 2\langle E^4 \rangle - 6\langle E^3 \rangle + 2\langle E \rangle\langle E^2 \rangle + 6\langle L^2 \rangle.$$

Clearly, the two algebraically independent invariants equivalent to $X^{(3)}$ and $X^{(4)}$ in the trace notation are $\langle E\tilde{E}^2 \rangle + \langle \tilde{E}E^2 \rangle$ and $\langle E\tilde{E}E\tilde{E} \rangle$. Moreover, the cubic invariant piece $\langle E\tilde{E}^2 \rangle + \langle \tilde{E}E^2 \rangle$, which does not belong to $B(O(3); 1_{3\times 3})$, is completely eliminated from $Y^{(4)}$. We further define

$$Y = X^{(4)} + \tfrac{4}{3}C^{(1)}X^{(3)} + \tfrac{4}{9}(C^{(1)})^2 C^{[2]} - (2\langle E^4 \rangle - 6\langle E^3 \rangle + 2\langle E \rangle\langle E^2 \rangle + 6\langle L^2 \rangle). \qquad (24)$$

Then,

$$Y = -2\langle E\tilde{E}E\tilde{E}\rangle.$$

This preferred labelling operator is the *unique* (up to elements of $B(O(3); 1_{3\times 3}) \cap Z(GL(3); O(3))$ and complex scalar multiples) linear combination over $Z(GL(3); O(3))$ of the invariants of [11] at order 4, which belongs to the $O(3)$-invariant Bethe subalgebra $B(O(3); 1_{3\times 3})$.

In this letter, we have pointed out that the 'missing label' in the $GL(3) \supset O(3)$ group reduction can be identified with the appropriate generator of the $O(3)$-invariant Bethe subalgebra of the twisted Yangian in the $GL(3)$ enveloping algebra. This identification answers the longstanding puzzle of the lack of any systematic way to resolve the labelling problem, and casts light on known results, such as Racah's proof ([12], cited in [11]) that there is *no* choice of Hermitian labelling operator with a rational spectrum. In general terms, it provides an interesting insight into conventional group representation theory, coming ultimately from the study of integrable systems in physics (see [3]).

Our present result has potentially wide applicability. Clearly, labelling problems exist whenever group theoretical considerations arise in the analysis of quantum systems, and 'missing' labels are the generic case. For concrete applications see the review [12] or for example [5, 6] in the context of configuration mixing in the nuclear shell model. As to generalizations, it is known, for example, that higher dimensional analogues of the $N = 3$ case have quadratically growing numbers of 'missing labels', for example two for $GL(4) \supset O(4)$, four for $GL(5) \supset O(5)$ and so on [8]. It is tempting to conjecture that also in these cases, the invariant Bethe subalgebra of the twisted Yangian will provide a sufficient set of commuting labelling operators. Along these lines one can further extend the analysis to other labelling problems, for example $Sp(4) \supset Sp(2)$, or even to exceptional embeddings such as that of $G_2 \supset SO(3)$ mentioned in the introduction. For the $U(3) \supset O(3)$ labelling problem itself, there is of course the task of numerical evaluation and analysis of the spectrum of preferred missing label (24) in the light of the present framework. Further work along these lines is in progress.

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